The Quantum Theoretical Harmonic Oscillator: Mathematical Structures

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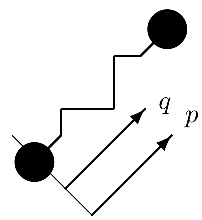
July 12th of 1997

The following text develops a new method of solving the eigenvalue problem of the quantum mechanical harmonic oscillator. Thereby are introduced new notions, like that of a reduced basis or that of a huge algebra. Furthermore are presented formulas for iterated commutation relations.

Commutation Relation and Hamilton Function

Oscillatory properties of a diatomic molecule, a carbon-monoxide molecule for example, can be modelled by a linear elastic oscillator. The observables for describing the oscillatory states of the oscillator are taken to be the dislocation from the equilibrium state and the relative momentum of the two masses.



The two observables "dislocation" q and "relative momentum" p are taken to be elements of an associative unital algebra, in other words: The elements can be added, multiplied and they can be multiplied with a scalar number. Especially in quantum theory the multiplication of the algebra elements is noncommutative because the dislocation- and momentum-elements are required to satisfy the Heisenberg commutation relation:

$$q\,p - p\,q = i\,1$$

(A commutation relation fully specifies the noncommutativity of the algebra elements on the left side of the equation.) In quantum theory complex numbers are used, the imaginary unit i above indicates this.

The oscillatory energy of a diatomic molecule is composed of a kinetic and a potential part:

$$H = \tfrac{1}{2}\,(p^2 + q^2)$$

The quadratic potential is the simplest of the polynomial kind that accounts for a stable equilibrium. As a function of dislocation and momentum the oscillatory energy is called Hamilton function. The commutation relation is normed in units of Planck's constant and the Hamilton function is normed in units of the natural frequency of the oscillator times Planck's constant, in that notation the following calculations become clearer. For the same reason, the calculations are done in linear combinations of the basic entities:

$$\left\{ \begin{array}{l} b := \frac{1}{\sqrt{2}}\,(q + ip), \\ a := \frac{1}{\sqrt{2}}\,(q - ip) \end{array} \right\} \quad \Rightarrow \quad \begin{array}{l} b\,a - a\,b = 1 \\ H = ab + \frac{1}{2}\,1 \end{array}$$

Next, wonder how to write the terms above as elements of an explicitly given associative algebra over the complex numbers?

Associative Algebras in Small Representation

In the following, algebras are called small represented, if their elements can be written as linear combinations of an explicitly given algebraic basis.

Above, concatenating symbols described an algebra multiplication. Associativity, for example seen in $a\,(b\,a) = (a\,b)\,a$, means that the parenthezising is irrelevant and need not be written down. Noncommutativity, for example $a\,b$ differing from $b\,a$, means that the order of the symbols is to be respected. In cases like $a\,a\,a$, where symbols are the same, the notation is shortened to a^3, thereby defining a notation of powers. Concatenation of the basic symbols, under these rules, gives a set on which the concatenation is an associative noncommutative operation. Such a structure is called a free monoid generated by the given symbols:

$$\mathrm{Mo}(\{a, b\}) = \{1, a, b, a^2, a\,b, b\,a, b^2, a^3, \ldots\}$$

Thereby the much needed symbolic multiplication is characterized by an algebraic structure.

Associative Algebra, Small, without Relations

The set $A_{\mathbb{C}} := \mathrm{span}_{\mathbb{C}}(\mathrm{Mo}(\{a, b\})) =$

$$= \left\{ \sum_{\alpha \in \mathrm{Mo}(\{a,b\})} x_\alpha\, \alpha \;\middle|\; (x_\alpha)_{\alpha \in \mathrm{Mo}(\{a,b\})} \in \mathbb{C}^{(\mathrm{Mo}(\{a,b\}))} \right\}$$

of all linear combinations with complex numbers and elements of the free monoid is a vector space. Addition and scalar multiplication are defined monoid-element-wise, as needed for example to write down the Hamilton function. (The symbol "span" denotes the set of all linear combinations, whereas $\mathbb{C}^{(\mathrm{Mo}(\{a,b\}))}$ describes the set of all families $(x_\alpha)_{\alpha \in \mathrm{Mo}(\{a,b\})}$ of coordinates, which have only finitely many nonzero coordinates.) Then, the free monoid is an algebraic basis of the complex vector space.

The multiplication, given by the monoid, makes that vector space an associative unital algebra; but therein the Heisenberg commutation relation is not valid, because the element $ba - ab$ is not the neutral element of the monoid multiplication.

Algebra, Small, with Heisenberg Commutation Relation

Starting from the vector space structure given above can be defined a new vector space, in which the commutation relation is valid. Inside the vector space $\mathrm{span}_{\mathbb{C}}(\mathrm{Mo}(\{a, b\}))$, the multiplicatively enveloped term $b\,a - a\,b - 1$ of the commutation relation spans a vector subspace I:

$$I := \mathrm{span}_{\mathbb{C}}(A_{\mathbb{C}}\,(b\,a - a\,b - 1)\,A_{\mathbb{C}})$$

Strikingly, elements of such a subspace can be multiplied from left or right with algebra elements; the result remains in the subspace. That property is also called stability with respect to left- and right multiplication. A vector subspace that is stable with respect to left- and right multiplication is also called ideal.

Consider an algebra element x (also called representative) and add it to each element of the ideal I, the resulting subset $x + I \subset A_{\mathbb{C}}$ is called coset and generally has no structure of a vector space.

Considering the set of all these subsets, namely cosets, can be shown: The set of all cosets partitions the algebra in nonoverlapping subsets (and each element of the algebra is in exactly one coset). On this set

$$A_{\mathbb{C}}/I := \{x + I |\ x \in A_{\mathbb{C}}\}$$

of subsets an addition, a scalar multiplication and an algebra multiplication can be found, by adding, scalar-multiplying and algebra-multiplying representatives of the cosets. That gives the set $A_{\mathbb{C}}/I$ of all cosets the structure of an algebra, a so-called quotient algebra. The crucial property of that construction is: Since the term of the Heisenberg commutation relation is an element of the ideal, representatives of cosets can be changed by simply adding this term, or some other multiplicatively enveloped version, to the representative:

$$b\,a - a\,b + I = b\,a - a\,b - (b\,a - a\,b - 1) + I = 1 + I$$

So, in this quotient algebra can be formulated the Heisenberg relation and the Hamilton function. Since the Heisenberg relation $b\,a - a\,b = 1$ is a linear combination of monoid elements, the set of all monoid elements cannot be an algebraic basis of that algebra.

The following reasoning indicates such an algebraic basis: A total order $a < b$ on the set of the generating elements gives a lexical order $1 < a < a\,a < a\,b < \ldots < b < b\,a < b\,b, \ldots$ on the elements of the free monoid. Distinguished elements therein are the normally ordered ones, like 1, a, b, $a\,a$, $a\,b$ or b^2. Applying the Heisenberg relation to nonnormally ordered monoid elements can produce linear combinations of normally ordered monoid elements:

$$b\,a + I = b\,a - (b\,a - a\,b - 1) + I = a\,b + 1 + I$$

(Parts of that argument are systematically presented in the Poincaré-Birkhoff-Witt-Theorem [DIX].). Thus, the subset of all normally ordered elements

$$\mathrm{PBW}\,(\{a, b\}) := \{1, a, b, a^2, a\,b, b^2, a^3, \ldots\} \quad \subset \mathrm{Mo}(\{a, b\})$$

of the free monoid is an algebraic basis of the quotient algebra, so each element therein can be written as a linear combination

$$x \in A_{\mathbb{C}}/I \Leftrightarrow \exists\, (x_{k,l})_{(k,l)\in\mathbb{N}_0\times\mathbb{N}_0} \left((x_{k,l})_{(k,l)\in\mathbb{N}_0\times\mathbb{N}_0} \in \mathbb{C}^{(\mathbb{N}_0\times\mathbb{N}_0)} \text{ and } x = \sum_{k,l\in\mathbb{N}_0} x_{k,l}\, a^k\, b^l + I \right)$$

of normally ordered monoid elements; plus an ideal, which contains all reformulations that can be done using the Heisenberg relation. The ideal often is omitted if no confusion arises.

Excursion: Lie Algebras

In each associative algebra A over a field $I\!K$ can be defined a commutator operation $A \times A \to A$, which assigns two elements their commutator:

$$\forall x, y\, (x, y \in A \Rightarrow [x, y] := x\,y - y\,x)$$

Usually a commutator is written down using a pair of brackets. The commutator operation is nonassociative, but fully characterizable by antisymmetry and the Jacobi Identity, except for specific relations like the Heisenberg relation (that is the main result of the Poincaré-Birkhoff-Witt-Theorem). Algebras, that have an antisymmetric multiplication satisfying the Jacobi Identity, are called Lie algebras. In considering the vector space of an associative algebra together with the commutator operation, one has a Lie algebra:

$$(A, +, I\!K, [., .])$$

The special Lie algebra $(A_{\mathbb{C}}/I, +, I\!K, [., .])$ has two Lie subalgebras: Firstly the Heisenberg Lie algebra

$$(\mathrm{span}_{\mathbb{C}}(\{1, a, b\})\,, +, \mathbb{C}, [b, a] = 1)$$

and the oscillator Lie algebra

$$(\mathrm{span}_{\mathbb{C}}(\{1, a, b, H\})\,, +, \mathbb{C}, [b, a] = 1, [b, H] = b, [a, H] = -a).$$

Eigenvalues of the Hamilton Function

In going from an energy function of monoid elements to actual numbers, formulate a so-called eigenvalue problem:

$$H\, z_\lambda = \lambda\, z_\lambda$$

Observe that both, the eigenvalues λ and the eigenelements z_λ are unknown.

An elementary transformation $(H - \lambda\, 1)\, z_\lambda = 0$, identifies the eigenelements as zero divisors. And the factor $(H - \lambda\, 1)$ cannot be invertible for an eigenvalue λ. (No element in an associative algebra can be a zero divisor and invertible at the same time.)

Therefore assuming invertibility lets identify possible energy eigenvalues by spotting problems with definedness. The rest is calculation:

$$\begin{aligned}
1 &= (H - \lambda\, 1)\, x_\lambda = (a\, b - \underbrace{(\lambda - \tfrac{1}{2})}_{=:\overline{\lambda}}\, 1) \sum_{k,l \in \mathbb{N}_0} x_{k,l}\, a^k\, b^l = \\
&= \sum_{k,l \in \mathbb{N}_0} x_{k,l}\, (a\, b\, a^k\, b^l - \overline{\lambda}\, a^k\, b^l) = \\
&= \sum_{k',l' \in \mathbb{N}_0} x_{k',l'}\, a^{k'+1}\, b^{l'+1} - \sum_{k,l \in \mathbb{N}_0} x_{k,l}\, (\overline{\lambda} - k)\, a^k\, b^l
\end{aligned}$$

The above follows from normally ordering the monoid elements using the iterated Heisenberg commutation relation:

$$b\,a - a\,b = 1 \quad \Rightarrow \quad b\,a^k - a^k\,b = k\,a^{k-1}$$

Changing summation indices ($k := k'+1$, $l := l'+1$) makes monoid elements comparable

$$1 = \sum_{k,l\geq 1} x_{k-1,l-1} a^k\,b^l - \sum_{k,l\geq 1} x_{k,l}\,(\overline{\lambda}-k) a^k\,b^l - \sum_{k\geq 1, l=0} \ldots - \sum_{k=0, l\geq 1} \ldots - \sum_{k=0,l=0} \ldots$$

and coordinates of same monoid elements can be gathered:

$$1 = \sum_{k,l\geq 1} \underbrace{(x_{k-1,l-1} - x_{k,l}(\overline{\lambda}-k))}_{=0} a^k\,b^l - \sum_{k\geq 1} \underbrace{x_{k,0}\,(\overline{\lambda}-k)}_{=0} a^k - \sum_{l\geq 1} \underbrace{x_{0,l}\,\overline{\lambda}}_{=0} b^l - \underbrace{x_{0,0}\,\overline{\lambda}}_{=1} 1$$

Succeeded by comparing coefficients (this would require the algebraic freeness of the monoid elements). Defined coordinates emerge under the condition $\overline{\lambda}-k \neq 0$ as $x_{0,0} = -\frac{1}{\overline{\lambda}}$, $x_{k,0} = 0$ if $k \geq 1$, $x_{0,l} = 0$ if $l \geq 1$, giving the remaining coordinates to be $x_{k,l} = \frac{x_{k-1,l-1}}{(\overline{\lambda}-k)} = \ldots = \frac{x_{k-m,l-m}}{(\overline{\lambda}-k+m-1)\cdots(\overline{\lambda}-k)}$ and thus $x_{k,l} = \delta_{k,l}\,\frac{-1}{\overline{\lambda}\,(\overline{\lambda}-1)\cdots(\overline{\lambda}-k)}$. Then the potentially inverse element is:

$$x_\lambda = \sum_{n\in\mathbb{N}_0} \frac{-1}{\overline{\lambda}\,(\overline{\lambda}-1)\cdots(\overline{\lambda}-n)}\,a^n\,b^n$$

Those values λ (or $\overline{\lambda}$), that make the term above undefined, possibly are energy eigenvalues:

$$\overline{\lambda} \in \mathbb{N}_0 \qquad \Leftrightarrow \qquad \lambda \in \mathbb{N}_0 + \tfrac{1}{2}$$

Remarkably, the absorption spectrum of diatomic molecules [BÖHM] does not contradict the calculative result! But, since this partially defined inverse element is a sum of infinitely many summands, it cannot be an element of the small quotient algebra, which contains only elements that can be written as linear combinations (finitely many summands!) of normally ordered monoid elements.

Nevertheless the potentially inverse element is physically relevant, but it cannot be described with the standard mathematical notions (linear combination, algebraic freeness). At this point three problems may be recognizable:

$\sum_\infty$ The generalization of the notion of a linear combination to sums with infinitely many summands.

FREE The generalization of linear freeness (also called linear independence) to sums with infinitely many summands, to validate any comparison of coefficients,

MULT and a suitably extended algebra, so that multiplicative inverses are uniquely determined.

These problems are considered in the following sections.

Reduced Basis

A sum with infinitely many summands is best described by considering it to be a limit point. A limit point of a sequence (or a net) of points is an element with each of its neighborhoods containing almost all elements of the sequence (or net), except finitely many.

The notion of a neighborhood is given by subsets that are neighborhoods of all their elements, so-called open sets. The set of all open sets of an original set E is called topology $\mathcal{O} \subset \mathcal{P}(E)$, where $\mathcal{P}(E)$ is the set of all subsets of the original set E. The notion of a limit point becomes unique if the topology contains enough open sets, to satisfy the so-called Hausdorff property. In that case continuous functions are those that conserve limit processes, in other words: The image of a sequence (or a net) converges to the image of the limit point.

A vector space $(E, +, \mathbb{C})$ with a Hausdorff topology $\mathcal{O}$ and continuous addition and scalar multiplication is called Hausdorff topological vector space $(E, +, \mathbb{C}, \mathcal{O})$. That notion of a vector space permits to extend the notions of a linear combination and an algebraic basis.

Extending the notion of a linear combination of vectors from a set $A \subset E$, consider a limit of a net of linear combinations:

$$x \in E \text{ is called linear sum of the family } (x_a)_{a\in A} \in \mathbb{C}^A \quad :\Leftrightarrow \quad x = \lim_{F_{\text{endlich}} \subset A} \sum_{a\in F} x_a\, a$$

Then the element can be written $x = \sum_{a\in A} x_a\, a$. The set of all linear sums of a given generating set is called linear sum of the given set:

$$\text{A set } S \text{ is called Linear Sum of a set } A :\Leftrightarrow$$
$$S = \left\{ x \,\middle|\, x \in E \text{ and } \exists\, (x_a)_{a\in A} \left((x_a)_{a\in A} \in \mathbb{C}^A \text{ and } x = \sum_{a\in A} x_a\, a \right) \right\}$$

That linear sum, written $S := \text{sum}_{\mathbb{C},\mathcal{O}}(A)$, is a topological vector subspace because addition and scalar multiplication are continuous: That can be seen with $x, y \in \text{sum}_{\mathbb{C},\mathcal{O}}(A) \subset E$, $s \in \mathbb{C}$ and [TGIII.42 §5.5 Proposition 6]:

$x + s\, y = \lim_{F_{\text{finite}} \subset A} \sum_{a\in F} x_a\, a + s \lim_{G_{\text{finite}} \subset A} \sum_{a\in G} y_a\, a =$
$\lim_{M_{\text{finite}} \subset A} \sum_{a\in M} (x_a + s\, y_a)\, a \in \text{sum}_{\mathbb{C},\mathcal{O}}(A)$

Consider a first example of such a linear sum space: The space of all families of complex numbers over a given set, for example PBW $(\{a, b\})$, has the structure of a vector space $\left(\mathbb{C}^{\text{PBW}(\{a,b\})}, +, \mathbb{C}\right)$. Furnished with the so-called product topology, with respect to the topology of the absolute value on the complex numbers, that space is a Hausdorff topological vector space $\left(\mathbb{C}^{\text{PBW}(\{a,b\})}, +, \mathbb{C}, \mathcal{O}_\pi\right)$. By the product topology, each element of that space can be written as a linear sum of singly supported families of coordinates (so-called coordinate elements).

up to *in*finitely many

Thinking in monoid elements, the coordinate space depicts a vector space that can be written as a linear sum of the generating elements:

$$\left(\mathbb{C}^{\mathrm{PBW}(\{a,b\})},\, +,\, \mathbb{C},\, \mathcal{O}_\pi\right) \quad \overset{\text{vector space}}{\cong} \quad \mathrm{sum}_{\mathbb{C},\,\mathcal{O}_\pi}\left(\mathrm{PBW}\left(\{a,b\}\right)\right)$$

Since, here, the generating elements are the normally ordered elements of the monoid, that has been used for the eigenvalue calculations, one can imagine the calculation done in that space. (Later is calculated the explicit form of the associative multiplication which is based on the Heisenberg commutation relation.) To validate the comparison of coefficients, a new notion of freeness is needed: $\sum_\infty$

$$A \subset E \text{ is called reduced free } :\Leftrightarrow$$

$$\forall\, (x_a)_{a\in A} \left(\left((x_a)_{a\in A} \in \mathbb{K}^A \text{ and } \sum_{a\in A} x_a\, a = 0\right) \Rightarrow (x_a)_{a\in A} = (0)_{a\in A}\right)$$

Especially, index sets of product topologized (product) spaces are reduced free which allows to compare coefficients of linear sums as indicated above. Reduced free sets are also algebraically free, as can be seen by restricting one's scope to linear sums with only finitely many summands. The notions of a linear sum space and that of reduced freeness together give that of a reduced basis $B \subset E$: FREE

$$B \text{ is called reduced basis} \quad :\Leftrightarrow \quad E \subset \mathrm{sum}_{\mathbb{K},\,\mathcal{O}}(B) \text{ and } B \text{ is reduced free.}$$

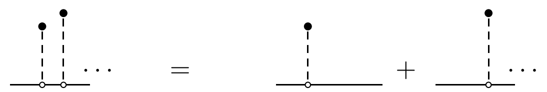
Since a reduced basis is always algebraically free, the extension theorem [AII.95 §7.1 Theorem 2] furnishes an algebraic basis of the Hausdorff topological vector space. Then, the reduced basis is a subset of the algebraic basis, which motivates the name "reduced basis". Remarkably, sums with infinitely many summands of a reduced basis cover the entire vector space with less basis elements than the algebraic basis. For that task, having only linear combinations, an algebraic basis needs more basis elements.

Examples for a Reduced Basis

The following is a list of spaces, each having a reduced basis:

The rational numbers, together with the topology that is given by the absolute value, make up an Hausdorff topological vector space $\left(\mathbb{Q},\, +,\, \mathbb{Q},\, \mathcal{O}_{|.|}\right)$ with the reduced basis $B = \{1\}$. This Hausdorff topological vector space is not complete. Thus, a reduced basis does not guarantee completeness.

In direct sum spaces $\left(\mathbb{K}^{(B)},\, +,\, \mathbb{K},\, \mathcal{O}_{\mathrm{box}}|_{\mathbb{K}^{(B)}}\right)$, that are inducedly box topologized, the injection of the index set B is both, an algebraic basis and a reduced basis. Due to the many neighborhoods (in slight analogy to a discrete topology) each limit point of different coordinate elements needs to be reached in finitely many steps, that means: Linear sums of coordinate elements with respect to a box topology can have only finitely many summands.



In each Hilbert space a maximal orthonormal system is a reduced basis of the Hilbert space.

The last example shows also an algebra in large representation [AIII.27 §2.10]. The free monoid $\mathrm{Mo}(X)$ of an arbitrarily chosen set X gives an associative multiplication and is, as a set, the reduced basis of the linear sum space:

$$\left(\mathbb{K}^{\mathrm{Mo}(X)},\,+,\,\mathbb{K},\,\cdot,\,\mathcal{O}_\pi\right) \overset{\text{associative algebra}}{\cong} \mathrm{sum}_{\mathbb{K},\,\mathcal{O}_\pi}(\mathrm{Mo}(X))$$

$$\begin{aligned}\left(\sum_{\alpha\in\mathrm{Mo}(X)} x_\alpha\,\alpha\right)\left(\sum_{\beta\in\mathrm{Mo}(X)} y_\beta\,\beta\right) &= \sum_{\alpha,\beta\in\mathrm{Mo}(X)} x_\alpha\,y_\beta\,\alpha\,\beta = \\ &= \underbrace{\sum_{\gamma\in\mathrm{Mo}(X)} \underbrace{\left(\sum_{\alpha,\beta\in\mathrm{Mo}(X)\text{ and }\gamma=\alpha\,\beta} x_\alpha\,y_\beta\right)}_{\text{finitely many}}\gamma}_{\text{up to infinitely many}}\end{aligned}$$

In algebras of large representation, elements can be described by linear sums, whereas the coordinates of a product of algebra elements have to be sums of finitely many summands. (In algebras of small representation, all elements can be written as linear combinations, there the coordinates of products are automatically sums of finitely many summands.)

Algebraic-, Reduced-, Topological Basis

In mathematics there is a third related basis notion, namely that of a topological basis $B \subset E$ of a Hausdorff topological vector space $(E,\,+,\,\mathbb{K},\,\mathcal{O})$:

$$\begin{gathered} B \text{ is a topological basis of } E :\Leftrightarrow \\ E \subset \overline{\mathrm{span}_{\mathbb{K}}(B)} \text{ and } \forall b\left(b\in B \Rightarrow b \notin \overline{\mathrm{span}_{\mathbb{K}}(B\setminus\{b\})}\right)\end{gathered}$$

The line over a set denotes the smallest superset having an open complement, which defines the so-called topological closure of a set. (A topologically closed set contains each element of its border, an open set none, because an open set is the neighborhood of all its elements.) There is an example which shows that all three notions of a basis are fundamentally different:

Therefore consider the set $\mathcal{B}([0,1],\mathbb{R}) \subset \mathbb{R}^{[0,1]}$ of all bounded real-valued functions on the unit interval $[0,1] \subset \mathbb{R}$:

$$(\mathcal{B}([0,1],\mathbb{R}),\,+,\,\mathbb{R},\,\|.\|)$$

This set, with pointwise addition and scalar multiplication, becomes a vector space. The unsigned maximal value of each function defines a norm on that vector space, and thereby makes it a Hausdorff topological vector space.

The set of all monomial functions

$$B := \{m_n : [0,1] \to \mathbb{R}, x \mapsto x^n |\ n \in \mathbb{N}_0\}$$

spans the vector space $\operatorname{span}_{\mathbb{R}}(B)$ of all polynomial functions and is an algebraic basis therein.

The linear sum $\operatorname{sum}_{\mathbb{R}\ \mathcal{O}_{\|\cdot\|}}(B)$ of the set of all monomial functions is the set of all functions with a power series with respect to the zero convergent on the unit interval. This space is a Hausdorff topological subspace and the uniqueness of the Taylor expansion makes the set of the monomial functions reduced free. So the set of all monomial functions is a reduced basis of this vector space.

By the Theorem of Stone and Weierstraß [TGX §4.1 Theorem 2], the norm topological closure $\overline{\operatorname{span}_{\mathbb{R}}(B)}$ of the set of all polynomial functions is the set of all continuous functions. Furthermore the set B of all monomial functions is not topologically free, because [TGX §4.2 Lemma 2] shows, that the identical function can be approximated uniformly by a sequence of monomial functions of even exponent.

Thus, generally, a topological basis and a reduced basis are something different, the same is true when comparing them with an algebraic basis. This closes the introduction of the reduced basis.

Heisenberg Multiplication

The previously constructed associative algebra $\operatorname{span}_{\mathbb{C}}(\mathrm{PBW}(\{a,b\}))$ of small representation, has a multiplication that is based on the Heisenberg commutation relation. In the following, that associative multiplication is made explicit by multiplying two linear combinations that represent algebra elements x, y:

$$x\,y = \left(\sum_{r,s\in\mathbb{N}_0} x_{r,s}\, a^r\, b^s\right)\left(\sum_{t,u\in\mathbb{N}_0} y_{t,u}\, a^t\, b^u\right) = \sum_{r,s,t,u\in\mathbb{N}_0} x_{r,s}\, y_{t,u}\, a^r\, b^s\, a^t\, b^u$$

The iterated commutator relation

$$b\,a - a\,b = 1 \qquad \Rightarrow \qquad b^s\, a^t = \sum_{m=0}^{\min(s,t)} \frac{s!\,t!}{(t-m)!\,m!\,(s-m)!}\, a^{t-m}\, b^{s-m}$$

reestablishes the normal order. And a change of summation indices

$$\sum_{r,s,t,u\in\mathbb{N}_0} \sum_{m=0}^{\min(s,t)} S(r,s,t,u,m) = \sum_{k,l,n\in\mathbb{N}_0} \sum_{q=0}^{k} \sum_{p=0}^{l} S(k-q, n+p, n+q, l-p, n)$$

collects the coefficients of same monoid elements:

$$x\,y = \sum_{k,l\in\mathbb{N}_0} \left(\sum_{n\in\mathbb{N}_0} \sum_{q=0}^{k} \sum_{p=0}^{l} \frac{(n+p)!\,(n+q)!}{p!\,n!\,q!}\, x_{k-q,n+p}\, y_{n+q,l-p}\right) a^k\, b^l$$

All the sums above have finitely many summands, because linear combinations are used. As can be seen in the following example, the multiplication cannot be extended onto the linear sum space $\mathrm{sum}_{\mathbb{C},\,\mathcal{O}_\pi}(\mathrm{PBW}(\{a,b\}))$; because two elements, with all coordinates equal to one, have at least one undefined product coordinate:

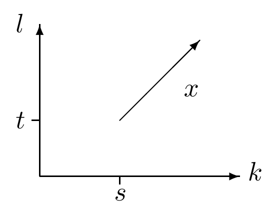
$$\left(\left(\sum_{k,l\in\mathbb{N}_0} 1\, a^k\, b^l\right)^2\right)_{0,0} = \sum_{n\in\mathbb{N}_0} n!$$

An immediate question may be how to restrict the linear sum space appropriately to make the multiplication defined?

Diagonal Algebra in Large Representation

The first possibility give so-called diagonal elements:

$$D := \left\{ \sum_{n\in\mathbb{N}_0} x_n\, a^{s+n}\, b^{t+n} \,\middle|\, (x_n)_{n\in\mathbb{N}_0} \in \mathbb{C}^{\mathbb{N}_0} \text{ and } s,t\in\mathbb{N}_0 \right\}$$



The multiplication, that is given by the Heisenberg commutation relation, is defined on the linear span $\mathrm{span}_{\mathbb{C}}(D)$, the space of all linear combination of diagonal elements. This can be seen by calculatively verifying the relation $D\,D \subset D$. MULT
That large algebra is associative, it has the "one" as the multiplicative neutral and is called diagonal algebra. And this is the algebra in which the energy eigenproblem of the harmonic oscillator can be solved. Apart from the eigenvalues can be found the eigenelements and the well-known creation- and annihilation relations:

$$H\, z_n = (n + \tfrac{1}{2})\, z_n \qquad n \in \mathbb{N}_0$$

$$z_n = c_n \sum_{s\in\mathbb{N}_0} \frac{(-1)^s}{s!}\, a^{n+s}\, b^s \qquad c_n \in \mathbb{C}\setminus\{0\}$$

$$a\, z_n = z_{n+1} \qquad \text{creation relation for } c_n = c_{n+1}$$

$$b\, z_{n+1} = (n+1)\, z_n \qquad \text{annihilation relation for } c_n = c_{n+1}$$

The notions of a trace relation $\mathrm{Tr}(.)$ and an expectation relation $\langle . \rangle$ yield together with the, antilinearly and antimultiplicatively extended, star operation $^+$

$$\mathrm{Tr}\left(\sum_{k,l\in\mathbb{N}_0} x_{k,l}\, a^k\, b^l\right) = \sum_{n\in\mathbb{N}_0} x_{n,n}$$

$$\langle z \rangle_n = \mathrm{Tr}(z\, x_n\, x_n^+)$$
$$b^+ := a \text{ and } a^+ := b \text{ and } 1^+ := 1 \text{ with } (\mu\, \alpha\, \beta)^+ = \mu^*\, \beta^+\, \alpha^+$$
$$(\mu \in \mathbb{C}, \mu^* \text{ describes the complex conjugation }, \alpha, \beta \in \mathrm{PBW}\,(\{a,b\}))$$

the well-known expectation values

$$\langle q \rangle_n = \langle p \rangle_n = 0$$
$$\langle q^2 \rangle_n = \langle p^2 \rangle_n = (n + \tfrac{1}{2})\, \frac{|c_n|^2}{e}$$

for the relative momentum and the dislocation observables of the harmonic oscillator ($e = \exp(1) \in \mathbb{R}$). The Heisenberg uncertainty relation follows immediately:

$$\sqrt{\langle q^2 \rangle_n}\, \sqrt{\langle p^2 \rangle_n} = (n + \tfrac{1}{2})\, \frac{|c_n|^2}{e}.$$

Huge Contour Algebra

Another vector subspace is given by families of coordinates that have a contour:

$$R_{\mathbb{C}} = \left\{ x \,\middle|\, x = \sum_{k,l \in \mathbb{N}_0} x_{k,l}\, a^k\, b^l \in \mathrm{sum}_{\mathbb{C},\,\mathcal{O}_\pi}(\mathrm{PBW}\,(\{a,b\})) \text{ and } \exists c_x, d_x \left(c_x, d_x \in \mathbb{R}_0^+ \text{ and } |x_{k,l}| \le c_x\, \frac{d_x^{(k+l)}}{k!\, l!} \right) \right\}$$

That furnishes an associative algebra with a “one”. But, this algebra is not a small algebra, neither a large algebra, as can be seen by multiplying an element of a fixed contour to itself. Algebras, with a coordinate representation, in which the sums of coordinates after a multiplication can have infinitely many summands, are called algebras of huge representation, or huge algebras. In the huge algebra above, exponentiating elements of the Heisenberg Lie algebra

$$e^{s\,a}\, e^{t\,b}\, e^{s'\,a}\, e^{t'\,b} = e^{s'\,t\,1}\, e^{(s+s')\,a}\, e^{(t+t')\,b},$$

yields a parameterization of the Heisenberg Lie group with the group operation:

$$(r,s,t)\,(r',s',t') = (r + r' + s'\,t, s + s', t + t').$$

Analogously, only with much greater calculative work, the same is possible for the oscillator Lie algebra.

Oscillator Multiplication and -Group

Being able to associatively multiply elements of the oscillator Lie algebra, so that exponential functions of these elements are defined, consider a complex associative algebra, with a multiplication that is given by a free monoid, which is generated by the symbols a, b and h. And where, supplementing the Heisenberg commutation relation, there are two other commutation relations, which result from taking $s = \pm 1$, considering the symbol h as the Hamilton function and substituting the symbols a und b for c:

$$[h, c] = sc \Rightarrow \quad h^m c^n = \sum_{r=0}^{m} (sn)^r \binom{m}{r} c^n h^{m-r} \quad (m, n \in \mathbb{N}_0, n \neq 0)$$

The changes of summation indices needed are

(i) $$\sum_{r=0}^{\infty}\sum_{u=0}^{\infty} S(r,u) = \sum_{m=0}^{\infty}\sum_{j=0}^{m} S(j, m-j)$$

(ii) $$\sum_{u=0}^{\infty}\sum_{w=0}^{r} S(u,w) = \sum_{m=0}^{\infty}\sum_{s=0}^{\min(m,r)} S(m-s, s) \qquad (r \in \mathbb{N}_0)$$

(iii) $$\sum_{q=0}^{\infty}\sum_{t=1}^{\infty} S(q,t) = \sum_{l=1}^{\infty}\sum_{i=0}^{l-1} S(i, l-i)$$

(iv) $$\sum_{q=0}^{\infty}\sum_{m=0}^{\min(q,s)} S(q,m) = \sum_{l=0}^{\infty}\sum_{t=0}^{s} S(l+t, t) \qquad (s \in \mathbb{N}_0)$$

(v) $$\sum_{r=0}^{\infty}\sum_{u=0}^{\infty}\sum_{v=0}^{r}\sum_{w=0}^{r-v} S(r,u,v,w) =$$
$$\sum_{m=0}^{\infty}\sum_{x=0}^{\infty}\sum_{y=0}^{x}\sum_{z=0}^{\min(m,y)} S(x, m-z, x-y, y-z)$$

(vi) $$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\sum_{m=0}^{\min(q,s)} S(p,q,s,t,m) =$$
$$= \sum_{k=0}^{\infty}\sum_{l=1}^{\infty}\sum_{j=0}^{l-1}\left(\left(\sum_{i=1}^{k} S(k-i, j, i, l-j, 0)\right) + \right.$$
$$\left. + \left(\sum_{n=1}^{\infty}\sum_{i=0}^{k} S(k-i, j+n, i+n, l-j, n)\right)\right)$$

That yields the associative multiplication of two algebra elements x and y to be:

$$x\,y = \sum_{k,m\in\mathbb{N}_0} (c_0(k,0,m) + c_2(k,0,m))\, a^k h^m +$$
$$+ \sum_{k,l,m\in\mathbb{N}_0, l\neq 0} (c_0(k,l,m) + c_1(k,l,m) + c_2(k,l,m) + c_3(k,l,m))\, a^k b^l h^m$$

Where the coefficients are given by:

$$c_0(k,l,m) = \sum_{j=0}^{m} x(k,l,j)\, y(0,0,m-j)$$

$$c_1(k,l,m) = \sum_{r=0}^{\infty}\sum_{i=0}^{l-1}\sum_{j=0}^{\min(m,r)} (i-l)^{r-j}\binom{r}{j}\, x(k,i,r)\, y(0,l-i,m-j)$$

$$c_2(k,l,m) = \sum_{r=0}^{\infty}\sum_{s=1}^{\infty}\sum_{i=0}^{\min(k,s)}\sum_{j=0}^{\min(m,r)} \frac{s^{r-j}\, s!}{i!}\binom{r}{j}\binom{l-i+s}{l}\cdot$$
$$\cdot\, x(k-i,l-i+s,r)\, y(s,0,m-j)$$

$$c_3(k,l,m) = \sum_{x=0}^{\infty}\sum_{y=0}^{x}\sum_{z=0}^{\min(m,y)} \frac{x!}{(x-y)!\,(y-z)!\,z!}\cdot$$
$$\cdot\left(\sum_{j=0}^{l-1}(j-l)^{y-z}\left(\left(\sum_{i=1}^{k} i^{x-y}\, x(k-i,j,x)\, y(i,l-j,m-z)\right)+\right.\right.$$
$$+\left(\sum_{n=1}^{\infty}\sum_{i=0}^{k}\frac{(i+n)^{x-y}\,(i+n)!\,(j+n)!}{i!\,j!\,n!}\cdot\right.$$
$$\left.\left.\left.\cdot x(k-i,j+n,x)\, y(i+n,l-j,m-z)\right)\right)\right)$$

And a parameterization of the oscillator group has the following group operation:

$$(p,x,y,t)\,(p',x',y',t') = (p+p'+yx'e^{st}, x+x'e^{st}, y+y'e^{-st}, t+t'),$$

with p, x, y, t, p', x', y', $t' \in \mathbb{C}$.

Glossary

Family A collection of indexed elements, also interpretable as a function. (The indices are taken as arguments and the indexed elements are taken as images.)

Field The real numbers $\mathbb{R}$, the rational numbers $\mathbb{Q}$ and the complex numbers $\mathbb{C}$ are examples of fields.

Linear Span Set of all linear combinations of a given set.

Coordinate element Short term for an element in a product space, that has only one nonzero coordinate.

Topology According to the mathematical definition: A special kind of subset of the set of all subsets of a given original set. More intuitionable: The set of all those sets that have no border, or the sets which are the neighborhood of all their elements. Neighborhoods are the crucial notions in defining the limits of sequences and nets and the continuity of functions.

Hausdorff-Topology A kind of topology, that has enough open sets to make limits uniquely defined and to let continuous functions conserve limits.

Discrete Topology A topology on a set, that considers every subset to be an open set. This requires limits to be reached in finitely many steps, making the notion of a limit in the discrete topology a rather simple one.

Box-Topology Topology on a product space of topological sets. In such a topology, only those sums of differing coordinate elements are convergent, which have finitely many summands.

Product-Topology Topology on a product space of topological sets. In such a topology all sums of differing coordinate elements are convergent.

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(N.p. no place)